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The meniscus on a needle – a lesson in matching

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We reconsider the second approximation for the height of the meniscus on a slender vertical cylinder, which has been calculated by James using the method of matched asymptotic expansions. Fraenkel has warned that the asymptotic matching principle may fail in certain cases, and we confirm that failure here. This meniscus problem is used to discuss the kind of failures that may occur if Fraenkel's restricted form of the matching principle is not used. A less restricted matching principle is also suggested in this paper. With this matching principle, we show that the last known term of order $Re^3 \ln Re$ in the drag on a sphere in low-Reynolds-number flow is correct, even though it was found in a way that violates Fraenkel's warning.

1. Introduction

The method of matched asymptotic expansions is a powerful tool in solving singular perturbation problems of layer type. Since its formulation by Kaplun (1957), it has been used extensively in fluid mechanics as well as other fields. Van Dyke (1964) proposed a rule for matching (the asymptotic matching principle) that is particularly simple to use and has been applied successfully. However, Fraenkel (1969) warns that there are two cases in which the asymptotic matching principle may fail, the dangerous one being the case that the gauge functions in the asymptotic expansions are mixtures of powers and logarithms of the perturbation quantity ϵ. In that case, if the asymptotic matching principle is applied to an expansion that is truncated by separating terms that differ only by a logarithm of the perturbation quantity, an erroneous result may be obtained. A model example has been used in Fraenkel's paper to demonstrate this specific point, but so far we believe that only two physical problems have suffered from failure to heed the warning.

First, Crighton & Leppington (1973) used matched expansions to solve the problem of the diffraction of long-wavelength plane acoustic waves incident upon a thin semi-infinite plate. They found that a result that violated the reciprocity theorem would be obtained if they ignored Fraenkel's warning and applied the asymptotic matching principle to series truncated at terms in $\epsilon^2 \ln^2 \epsilon$ or $\epsilon^2 \ln \epsilon$. A consistent and correct result was found when the matching was done on the series truncated at terms in ϵ^2 as Fraenkel recommends.

Secondly, James (1974) used matched expansions to solve the problem of the height of a static meniscus outside a slender cylinder. He assumed that the outer expansion† for the height of the meniscus has the form

$$Z(R, \epsilon) = Z_1(R) + \epsilon^2 Z_2(R) + \dots,$$

† Here the outer expansion is the asymptotic expansion that is valid in the region far from the cylinder. The inner expansion is the asymptotic expansion that is valid in the region close to the cylinder. These definitions of outer and inner are exactly the opposite of those in James (1974).

and found that the inner expansion† has the form

$$z(r,\epsilon) = \ln\epsilon\, z_1(r) + z_2(r) + \epsilon^2 \ln^2\epsilon\, z_3(r) + \epsilon^2 \ln\epsilon\, z_4(r) + \epsilon^2 z_5(r) + \ldots,$$

where $R = \epsilon r$. Unaware of Fraenkel's warning, James truncated this series at z_3. By comparison with the numerical data of Huh & Scriven (1969), he concluded that the correction due to the term $\epsilon^2 \ln^2 \epsilon\, z_3(r)$ made the estimate worse than the first-order result. The second-order correction‡ is reconsidered in detail in the present work. Two interesting results are found.

First, as happened in the diffraction problem, the matching seems to be totally satisfactory when one applies the asymptotic matching principle in the truncated series, as James did, but it actually leads to a wrong $z_3(r)$, whereas the correct result is obtained when one heeds Fraenkel's warning.

Secondly, the matching should let us confirm the form of the asymptotic expansion (i.e. the gauge functions in the expansions). In what James did, the asymptotic matching principle seemed to confirm the form of the outer expansion as

$$Z(R,\epsilon) = Z_1(R) + \epsilon^2 Z_2(R) + \ldots.$$

However, the correct form is found to be

$$Z(R,\epsilon) = Z_1(R) + \epsilon^2 \ln\epsilon\, Z_2^*(R) + \epsilon^2 Z_2(R) + \ldots,$$

which falls out automatically when Fraenkel's restriction is imposed.

In their linear diffraction problem, Crighton & Leppington were fortunate in having a reciprocity principle to show when the asymptotic matching principle fails. In his nonlinear meniscus problem, James had numerical results to suggest that his partial second-order correction had the wrong sign. But there are problems where no such external check is available, yet it may be impracticable to carry the solution far enough to satisfy Fraenkel's restriction. For example, in the problem of flow past a sphere at low Reynolds number Re, the drag is found (Stokes 1851; Oseen 1910; Proudman & Pearson 1957; Chester & Breach 1969) to have the expansion

$$C_D = \frac{6\pi}{Re}\left[1 + \tfrac{3}{8} Re + \tfrac{9}{40} Re^2 \ln Re + \tfrac{9}{40}\left(\gamma + \tfrac{5}{3}\ln 2 - \tfrac{323}{360}\right) Re^2 + \tfrac{27}{80} Re^3 \ln Re + O(Re^3)\right], \tag{1.1}$$

where $\gamma = 0.5772$ is Euler's constant. Proudman & Pearson (1957) calculated only the first three terms of the above expansion, finding the term in $Re^2 \ln Re$ while dropping the more difficult term in Re^2. However, it happened that no mistake was made, for Chester & Breach (1969) confirmed the coefficient $\frac{9}{40}$ in extending the series. However, they have in turn violated Fraenkel's warning in retaining the term in $Re^3 \ln Re$ while dropping the one in Re^3 as being too difficult to calculate. This problem will be discussed in §4.2, and we will show that the term $Re^3 \ln Re$ in the drag is correct even though the Re^3 term is not found and Fraenkel's warning is not heeded.

2. Asymptotic matching principle

Matched asymptotic expansions is a technique for solving an initial- or boundary-value problem for a function $F(x,\epsilon)$ over a range of x in which no single asymptotic series for small ϵ can approximate F uniformly. Usually two asymptotic series are

† See footnote on preceding page.

‡ We count z_1 and z_2 as the first approximation, and z_3, z_4 and z_5 as the second-order correction.

needed to describe the function F in two different regions in the domain of interest. At least one of the two problems describing the series will not be well posed since the initial or boundary conditions outside that region are lost. One will have to use 'matching' in the overlap domain in order to recover those missing data. That makes the matching principle play an essential role.

Matching not only provides information on the undetermined constants in finding the asymptotic expansions, but also suggests (or checks) the appropriate gauge functions in the inner (or outer) series.

Van Dyke (1964) proposed an attractive and simple-to-use matching principle (the asymptotic matching principle) which says that the m-term inner expansion of (the n-term outer expansion) is equal to the n-term outer expansion of (the m-term inner expansion), where m and n can be any integers. In other words, if $Z(R,\epsilon)$ denotes the outer expansion with outer variable R, and $z(r,\epsilon)$ denotes the inner expansion with inner variable r, and $Z(R,\epsilon)$ is related to $z(r,\epsilon)$ by $Z(R,\epsilon) = f(\epsilon)\,z(r,\epsilon)$, one can introduce an operator O_n that truncates the asymptotic series for fixed outer variable R as $\epsilon \to 0$ up to and including the nth term, and an operator I_m that truncates the asymptotic series for fixed inner variable r as $\epsilon \to 0$ up to and including the mth term; then the asymptotic matching principle can be written as†

$$\mathrm{O}_n \mathrm{I}_m z(r) = \mathrm{I}_m \mathrm{O}_n \left[\frac{Z(R)}{f(\epsilon)}\right]. \tag{2.1}$$

The asymptotic matching principle is attractive because one can stop at any term and perform the matching without going into more complicated higher-order terms.

However, Fraenkel (1969) warns that, in spite of the beauty of the asymptotic matching principle, it may fail in the case that the gauge functions of the asymptotic expansions are mixtures of powers and logarithms of the perturbation quantity ϵ. He suggests that the matching should not be done by separating terms that differ by less than any power of ϵ. In other words, if Θ_k denotes an operator that truncates the asymptotic series for fixed outer variable R as $\epsilon \to 0$ up to and including all terms less than order $\epsilon^{k+\sigma}$ (σ is arbitrarily small) and Π_l denotes an operator that truncates the asymptotic series for fixed inner variable r as $\epsilon \to 0$ up to and including all terms less than order $\epsilon^{l+\sigma}$, Fraenkel proposes the restricted matching principle

$$\Theta_k \Pi_l z(r) = \Pi_l \Theta_k \left[\frac{Z(R)}{f(\epsilon)}\right], \tag{2.2}$$

in which l and k can be any real numbers.

We have seen that failure to heed Fraenkel's warning leads to error in some problems (Crighton & Leppington 1973) and not in others (Proudman & Pearson 1957). In §3 we will reconsider the problem of the meniscus on a needle using Fraenkel's restricted matching principle, and discuss what may happen if one ignores his warning.

3. The meniscus on a needle

The height of the static meniscus outside a slender cylinder was calculated by James (1974) using matched expansions. Being unaware of Fraenkel's warning, he found an erroneous correction term to the first approximation. We reconsider the whole second

† The notations O_n, I_m, Θ_k and Π_l that are used in this paper are the same as E_n, H_m, $E_{(k)}$ and $H_{(l)}$ that were used in Fraenkel (1969).

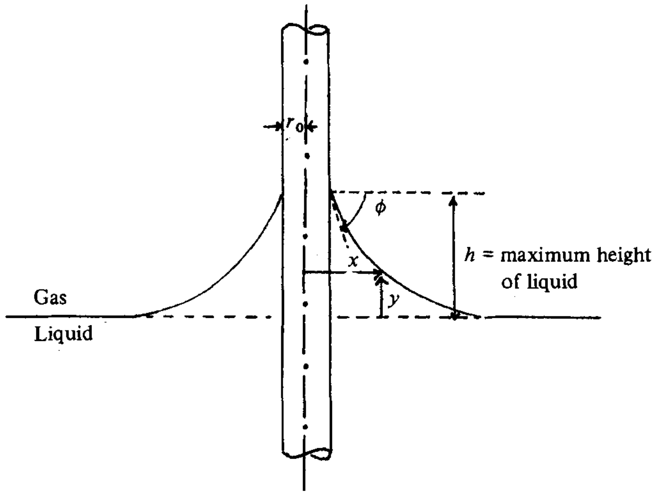

FIGURE 1. The static meniscus on a circular cylinder.

approximation with consideration of Fraenkel's warning. In order to make this paper self-contained, we will repeat the calculation of the first approximation. In concluding this section, the kinds of failure that may result from ignoring Fraenkel's warning will be discussed.

3.1. *Governing equations and boundary conditions*

The axisymmetric meniscus is sketched in figure 1. The liquid meets the solid surface at an angle ϕ, which is an experimentally determined value.

The Young–Laplace equation in the axisymmetric case can be written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]\left\{\frac{\rho g}{\sigma}y\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{1}{2}}-\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}\right\}, \tag{3.1}$$

with boundary conditions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan\phi \quad \text{at} \quad x = r_0,$$
$$y\rightarrow 0 \quad \text{as} \quad x\rightarrow\infty,$$

Here ρ is the density of the liquid, σ is the interfacial tension and g is the gravitational acceleration. There are two lengthscales in this problem: namely, the radius r_0 of the cylinder and the capillary length $l_c = (\sigma/\rho g)^{\frac{1}{2}}$.

We put (3.1) into dimensionless form in the following two ways. First, define $z = y/r_0$ and $r = x/r_0$, and it becomes

$$\frac{\mathrm{d}^2z}{\mathrm{d}r^2} = \left\{1+\left[\frac{\mathrm{d}z}{\mathrm{d}r}\right]^2\right\}\left\{\epsilon^2 z\left[1+\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right]^{\frac{1}{2}}-\frac{1}{r}\frac{\mathrm{d}z}{\mathrm{d}r}\right\}, \tag{3.2}$$

with $\epsilon = r_0/l_c$ (capillary number). Secondly, define $Z = y/r_0$ and $R = x/l_c = \epsilon r$, and it becomes

$$\frac{\mathrm{d}^2Z}{\mathrm{d}R^2} = \left\{1+\epsilon^2\left[\frac{\mathrm{d}Z}{\mathrm{d}R}\right]^2\right\}\left\{Z\left[1+\epsilon^2\left(\frac{\mathrm{d}Z}{\mathrm{d}R}\right)^2\right]^{\frac{1}{2}}-\frac{1}{R}\frac{\mathrm{d}Z}{\mathrm{d}R}\right\}. \tag{3.3}$$

We choose the capillary number ϵ to be the perturbation parameter. Because it is the ratio of two lengthscales, the problem is likely to be a singular perturbation (Van

Dyke 1964). This is also suggested by the fact that the last term in (3.2) becomes of the same order as the ϵ^2 term as $r \to 1/\epsilon^2$. This suggests that the perturbation expansion from (3.2) may be valid only in the region close to the cylinder and the rescaled differential equation (3.3) is needed for the distant region.

With these suggestions, r is chosen to be the inner variable. The inner solution $z(r)$ is the solution that will be valid in the region close to the cylinder. In other words, the differential equation governing the inner solution is (3.2), but now only the inner boundary condition, $z' = -\tan\phi$ at $r = 1$, is going to be enforced. $R = \epsilon r$ is chosen to be the outer variable. The outer solution $Z(R)$ is governed by (3.3) with the boundary condition $Z(R) \to 0$ as $R \to \infty$.

3.2. *First approximation*

3.2.1. *Outer expansion.* Assume the outer solution has the form

$$Z(R, \epsilon) = Z_1(R) + o(1).$$

From (3.3), $Z_1(R)$ is governed by the differential equation

$$Z_1'' + \frac{Z_1'}{R} - Z_1 = 0, \tag{3.4}$$

with boundary condition

$$Z_1(R) \to 0 \quad \text{as} \quad R \to \infty.$$

The general solution of (3.4) is

$$Z_1 = C_1' K_0(R) + C_1'' I_0(R),$$

where K_0 and I_0 are the two modified Bessel functions of zeroth order. The boundary condition shows that $C_1'' = 0$, so we have the first approximation

$$Z_1 = C_1' K_0(R) \tag{3.5}$$

with C_1' still undetermined.

3.2.2. *Inner expansion.* From the outer solution, K_0 behaves like a logarithm for small argument, which suggests that the first approximation in the inner expansion has the form

$$z(r, \epsilon) = z_1 \ln \epsilon + z_2 + o(1). \tag{3.6}$$

From (3.2), z_1 is governed by

$$z_1' = 0,$$

and it will automatically satisfy the boundary condition $z_1'(r = 1) = 0$. The solution is

$$z_1 = C_2. \tag{3.7}$$

From (3.2), z_2 is governed by

$$z_2'' = (1 + z_2'^2)\left(-\frac{z_2'}{r}\right), \tag{3.8}$$

with boundary condition $z_2' = -\tan\phi$ at $r = 1$. The general solution of (3.8) is

$$z_2 = C_1 + C_2'' \ln [r + (r^2 - C_2''^2)^{\frac{1}{2}}]. \tag{3.9}$$

The boundary condition gives $C_2'' = -c$, where $c = \sin\phi$. Hence $z^{(1)}$ can be written as†

$$z^{(1)} = C_2 \ln \epsilon + C_1 - c \ln [r + (r^2 - c^2)^{\frac{1}{2}}]. \tag{3.10}$$

† $z^{(1)}$ means the first approximation in $z(r)$.

3.2.3. *Matching in the first order.* From (3.5), one can use (2.2) with $m = n = 0$ and $f(\epsilon) = 1$

$$\Pi_0 \Theta_0 Z = -C_1' \ln \epsilon + (-C_1' \ln r - C_1' \gamma + C_1' \ln 2).$$

From (3.10) we find

$$\Theta_0 \Pi_0 z = C_2 \ln \epsilon + C_1 - c(\ln r + \ln z).$$

All the constants can be determined, and they are

$$C_1' = c,$$
$$C_2 = -C_1' = -c,$$
$$C_1 = c[2\ln 2 - \gamma].$$

$z(r, \epsilon)$ can be written as

$$z(r, \epsilon) = -c \ln \epsilon + c(2\ln 2 - \gamma) - c \ln [r + (r^2 - c^2)^{\frac{1}{2}}] + o(1). \tag{3.11}$$

The height H of the meniscus is equal to $z(r = 1)$:

$$H^{(1)} = z^{(1)}(r = 1) = \sin \phi \left\{ \ln \frac{4}{\epsilon(1 + \cos \phi)} - \gamma \right\}. \tag{3.12}$$

3.3. *Second approximation*

3.3.1. *Outer expansion.* The differential equation for the outer expansion (3.3) suggests that the correction to the first approximation will be of order ϵ^2, that is

$$Z(R, \epsilon) = Z_1(R) + \epsilon^2 Z_2(R) + o(\epsilon^2).$$

From (3.3), $Z_2(R)$ is governed by

$$Z_2'' + \frac{Z_2'}{R} - Z_2 = \left[\tfrac{3}{2} Z_1 - \frac{Z_1'}{R}\right] Z_1'^2, \tag{3.13}$$

with the boundary condition $Z_2(R) \to 0$ as $R \to \infty$.

The solution of (3.13) has the form

$$Z_2(R) = f_p(R) + C_2' f_h(R),$$

in which a particular solution $f_p(R)$ that vanishes at infinity can be found by variation of parameters. It consists of two complicated integrals. Since we need only the asymptotic form of $f_p(R)$ for matching, we can expand the integrands for small R and integrate term by term. The first few terms are

$$f_p(R) = \frac{1}{4} \frac{c^3}{R^2} - \frac{c^3}{4} \ln^2 R + \frac{c^3}{4} \ln R + \tfrac{3}{16} c^3 + \dots .$$

The homogeneous solution $f_h(R)$ that vanishes at infinity is a multiple of

$$f_h(R) = K_0(R),$$

so we have the second approximation

$$Z(R, \epsilon) = cK_0(R) + \epsilon^2 \left[\left(\frac{1}{4} \frac{c^3}{R^2} - \frac{c^3}{4} \ln^2 R + \frac{c^3}{4} \ln R + \tfrac{3}{16} c^3 + \dots \right) + C_2' K_0(R) \right] + o(\epsilon^2). \tag{3.14}$$

3.3.2. *Inner expansion.* The outer solution suggests correctly that the inner solution has the form

$$z(r, \epsilon) = \ln \epsilon \, z_1(r) + z_2(r) + \epsilon^2 \ln^2 \epsilon \, z_3(r) + \epsilon^2 \ln \epsilon \, z_4(r) + \epsilon^2 z_5(r) + \dots .$$

From (3.2), z_3 is governed by

$$z_3''+\left[\frac{1+3z_2'^2}{r}\right]z_3' = 0,$$

with the boundary condition $z_3'(r=1)=0$. The solution is

$$z_3(r) = C_3. \tag{3.15}$$

From (3.2), z_4 is governed by

$$z_4''+\left[\frac{r^2+2c^2}{r^2-c^2}\right]\frac{z_4'}{r} = -c\left[\frac{r^2}{r^2-c^2}\right]^{\frac{3}{2}},$$

with the boundary condition $z_4'(r=1)=0$. The general solution of the differential equation is

$$z_4(r) = C_4+C_5''\left\{-\frac{r}{(r^2-c^2)^{\frac{1}{2}}}+\ln[r+(r^2-c^2)^{\frac{1}{2}}]\right\}-\tfrac{1}{4}c^3\ln[r+(r^2-c^2)^{\frac{1}{2}}]-\tfrac{1}{4}cr(r^2-c^2)^{\frac{1}{2}},$$

and imposing the boundary condition gives

$$C_5'' = \tfrac{1}{2}c-\tfrac{1}{2}c^3.$$

Thus we have

$$z_4(r) = C_4+(\tfrac{1}{2}c-\tfrac{3}{4}c^3)\ln[r+(r^2-c^2)^{\frac{1}{2}}]-\tfrac{1}{4}cr(r^2-c^2)^{\frac{1}{2}}-(\tfrac{1}{2}c-\tfrac{1}{2}c^3)\frac{r}{(r^2-c^2)^{\frac{1}{2}}}. \tag{3.16}$$

From (3.2), z_5 is governed by

$$z_5''+\frac{r^2+2c^2}{r^2-c^2}\frac{z_5'}{r} = \left[\frac{r^2}{r^2-c^2}\right]^{\frac{3}{2}}c[(\ln 4-\gamma)-\ln(r+(r^2-c^2)^{\frac{1}{2}})],$$

with the boundary condition $z_5'(r=1)=0$. The solution is

$$\begin{aligned}z_5(r) &= \tfrac{1}{4}c(\ln 4-\gamma)[r(r^2-c^2)^{\frac{1}{2}}]-d[r(r^2-c^2)^{-\frac{1}{2}}]+\tfrac{1}{4}cr^2+C_5\\&\quad+[\tfrac{1}{4}c^3(\ln 4-\gamma)+d]\ln[r+(r^2-c^2)^{\frac{1}{2}}]-\tfrac{1}{4}cr(r^2-c^2)^{\frac{1}{2}}\ln[r+(r^2-c^2)^{\frac{1}{2}}]\\&\quad+\tfrac{1}{4}c^3r(r^2-c^2)^{-\frac{1}{2}}\ln[r+(r^2-c^2)^{\frac{1}{2}}]-\tfrac{1}{4}c^3\ln^2[r+(r^2-c^2)^{\frac{1}{2}}],\end{aligned} \tag{3.17}$$

in which

$$d = \tfrac{1}{4}c(2-c^2)\ln[1+(1-c^2)^{\frac{1}{2}}]-(1-c^2)\tfrac{1}{2}c(\ln 4-\gamma)-\tfrac{1}{4}c(1-c)^{\frac{1}{2}}. \tag{3.18}$$

Equations (3.11), (3.15)–(3.17) give the full second approximation:

$$\begin{aligned}z^{(2)}(r) &= -c\ln\epsilon+\{c(2\ln 2-\gamma)-c\ln[r+(r^2-c^2)^{\frac{1}{2}}]\}\\&\quad+C_3\epsilon^2\ln^2\epsilon+\epsilon^2\ln\epsilon\{C_4+(\tfrac{1}{2}c-\tfrac{3}{4}c^3)\ln[r+(r^2-c^2)^{\frac{1}{2}}]\\&\quad-\tfrac{1}{4}cr(r^2-c^2)^{\frac{1}{2}}-(\tfrac{1}{2}c-\tfrac{1}{2}c^3)\frac{r}{(r^2-c^2)^{\frac{1}{2}}}\Big\}\\&\quad+\epsilon^2\{\tfrac{1}{4}c(\ln 4-\gamma)\,r(r^2-c^2)^{\frac{1}{2}}-d[r(r^2-c^2)^{-\frac{1}{2}}]\\&\quad+\tfrac{1}{4}cr^2+C_5+[\tfrac{1}{4}c^3(\ln 4-\gamma)+d]\ln[r+(r^2-c^2)^{\frac{1}{2}}]\\&\quad-\tfrac{1}{4}cr(r^2-c^2)^{\frac{1}{2}}\ln[r+(r^2-c^2)^{\frac{1}{2}}]+\tfrac{1}{4}c^3\,r(r^2-c^2)^{-\frac{1}{2}}\ln[r+(r^2-c^2)^{\frac{1}{2}}]\\&\quad-\tfrac{1}{4}c^3\ln^2[r+(r^2-c^2)^{\frac{1}{2}}]\}.\end{aligned} \tag{3.19}$$

3.3.3. *Matching in the second order*. We apply Fraenkel's matching principle (2.2) with $m = n = 2$ and $f(\epsilon) = 1$. We have from (3.14)

$$\Pi_2\Theta_2 Z = -c\ln\epsilon + [-c\ln r - c\gamma + c\ln 2] + \frac{1}{4}\frac{c^3}{r^2} + \epsilon^2\ln^2\epsilon(-\tfrac{1}{4}c^3)$$
$$+\epsilon^2\ln\epsilon\{-\tfrac{1}{2}c^3\ln r - \tfrac{1}{4}cr + (\tfrac{1}{4}c^3 - C_2')\}$$
$$+\epsilon^2\{-\tfrac{1}{4}cr^2\ln r + (\tfrac{1}{4}c^3 - C_2')\ln r - \tfrac{1}{4}c^3\ln^2 r + \tfrac{1}{4}(\ln 2 + 1 - \gamma)\,cr^2$$
$$+\tfrac{3}{16}c^3 + C_2'(\ln z - \gamma)\},$$

and from (3.19),

$$\Theta_2\Pi_2 z = -c\ln\epsilon + [-c\ln r - c\gamma + c\ln 2] + \frac{1}{4}\frac{c^3}{r^2} + \epsilon^2\ln^2\epsilon C_3$$
$$+\epsilon^2\ln\epsilon\{(\tfrac{1}{2}c - \tfrac{3}{4}c^3)\ln r - \tfrac{1}{4}cr^2 + [C_4 + \ln 2(\tfrac{1}{2}c - \tfrac{3}{4}c^3) + \tfrac{5}{8}c^3 - \tfrac{1}{2}c\}$$
$$+\epsilon^2\{-\tfrac{1}{4}cr^2\ln r + [\tfrac{1}{4}c^3(\ln 4 - \gamma) + d + \tfrac{3}{8}c^3 - \tfrac{1}{2}c^3\ln 2]\ln r - \tfrac{1}{4}c^3\ln^2 r$$
$$+[\tfrac{1}{4}c(\ln 4 - \gamma) + \tfrac{1}{4}c - \tfrac{1}{4}c\ln 2]\,r^2 + [-\tfrac{1}{8}c^3(\ln 4 - \gamma) - d + C_5 + \tfrac{1}{16}c^3]$$
$$+\ln 2[\tfrac{1}{4}c^3(\ln 4 - \gamma) + d + \tfrac{3}{8}c^3] - \tfrac{1}{4}c^3\ln^2 2]\}.$$

The matching for $\epsilon^2\ln\epsilon$ is impossible because equating terms in $\epsilon^2\ln\epsilon\ln r$ gives a contradiction. This can be remedied by inserting a switchback† term in $\epsilon^2\ln\epsilon$ in the outer expansion. That means

$$Z(R,\epsilon) = Z_1(R) + \epsilon^2\ln\epsilon\, Z_2^*(R) + \epsilon^2 Z_2(R) + \dots.$$

From (3.3), Z_2^* is governed by

$$Z_2^{*\prime\prime} + \frac{Z_2^{*\prime}}{R} - Z_2^* = 0,$$

with boundary condition $Z_2^*(R)\to 0$ as $R\to\infty$, and the solution is $Z_2^* = C_2' K_0(R)$, so (3.14) should be

$$Z(R,\epsilon) = cK_0(R) + \epsilon^2\ln\epsilon[C_3' K_0(R)]$$
$$+\epsilon^2\left[\left(\frac{c^3}{4R^2} - \tfrac{1}{4}c^3\ln^2 R + \tfrac{1}{4}c^3\ln R + \tfrac{3}{16}c^3 + \dots\right) + C_2' K_0(R)\right] + o(\epsilon^2).$$

The correct form for matching is

$$\Pi_2\Theta_2 Z = -c\ln\epsilon + [-c\ln r - c\gamma + c\ln 2] + \frac{1}{4}\frac{c^3}{r^2} + \epsilon^2\ln^2\epsilon[-\tfrac{1}{4}c^3 - C_3']$$
$$+\epsilon^2\ln\epsilon\{-(\tfrac{1}{2}c^3 + C_3')\ln r - \tfrac{1}{4}cr^2 + [\tfrac{1}{4}c^3 - C_2' + (\ln 2 - \gamma)\,C_3']\}$$
$$+\epsilon^2\{c(-\tfrac{1}{4}r^2\ln r) + (\tfrac{1}{4}c^3 - C_2')\ln r - \tfrac{1}{4}c^3\ln^2 r + \tfrac{1}{4}(\ln 2 + 1 - \gamma)\,cr^2$$
$$+\tfrac{3}{16}c^3 + C_2'(\ln 2 - \gamma)\}.$$

† *The switchback phenomenon:* Guessing the appropriate form for the asymptotic expansion is often required in a perturbation solution. Sometimes an oversimplified form is assumed. Kaplun introduced the term 'switchback' to describe the situation when one tries to find a term of certain order, and is forced to reconsider a lower-order term. A detailed discussion of the switchback effect can be found in Lagerstrom & Casten (1972). The switchback term in our problem will be missed if Fraenkel's warning is ignored.

Now the matching can be carried out, giving

$$\left.\begin{aligned} C_2' = \tfrac{1}{4}c^3\gamma - d - \tfrac{1}{8}c^3, \quad C_3' = -\tfrac{1}{2}c + \tfrac{1}{4}c^3, \quad C_3 = \tfrac{1}{2}c(1-c^2) \\ C_4 = \tfrac{1}{2}c - \tfrac{1}{4}c^3 + d - \tfrac{1}{2}\gamma(c^3 - c) + \ln 2(c^3 - c), \\ C_5 = \tfrac{1}{4}c^3(\tfrac{1}{2} - \gamma^2) + d(1 - \ln 4 + \gamma) - \tfrac{1}{4}c^3 \ln^2 2 + (\tfrac{1}{2}\gamma c^3 - \tfrac{1}{4}c^3)\ln 2. \end{aligned}\right\} \quad (3.20)$$

The height of the meniscus up to $O(\epsilon^2)$ will be

$$H^{(2)} = z^{(2)}(r=1) = z_1(r=1)\ln\epsilon + z_2(r=1) + z_3(r=1)\,\epsilon^2\ln^2\epsilon + z_4(r=1)\,\epsilon^2\ln\epsilon + z_5(r=1)\,\epsilon^2, \quad (3.21)$$

where

$$z_1(r=1) = -\sin\phi,$$

$$z_2(r=1) = \sin\phi[\ln 4 - \ln(1+\cos\phi) - \gamma],$$

$$z_3(r=1) = \tfrac{1}{2}\sin\phi(1-\sin^2\phi),$$

$$z_4(r=1) = \sin\phi\{\cos^2\phi[\gamma + \tfrac{1}{4} + \ln\tfrac{1}{4}(1+\cos\phi)] + [\tfrac{1}{4} - \cos\phi]\},$$

$$\begin{aligned} z_5(r=1) = {} & \tfrac{1}{4}\sin\phi\cos\phi\,(\ln 4 - \gamma) - \frac{d}{\cos\phi} + \tfrac{1}{4}\sin\phi + d(1 - \ln 4 - \gamma) \\ & + \tfrac{1}{4}\sin^3\phi\,(\tfrac{1}{2} - \gamma^2 - \ln^2 2 + \gamma\ln 4 - \ln 2) \\ & + [\ln(1+\cos\phi)][\tfrac{1}{4}\sin^3\phi\left(\ln 4 - \gamma + \frac{1}{\cos\phi}\right) + d - \tfrac{1}{4}\sin\phi\cos\phi] \\ & - [\ln^2(1+\cos\phi)]\,\tfrac{1}{4}\sin^3\phi, \end{aligned}$$

and d is the expression in (3.18).

3.4. *Comparison with numerical results*

Our reconsideration of the second approximation was triggered by James' conclusion that, in comparison with numerical data, his correction due to the term $\epsilon^2\ln^2\epsilon z_3(r)$ makes the estimate worse than the first-order result. We compare our corrected result with the numerical data of Huh & Scriven (1969). The comparisons in the cases $\phi = 15°$, $30°$, $60°$ and $75°$ are plotted in figure 2.

We see that each of the three terms in the full second approximation successively improves the estimate.

3.5. *A lesson in asymptotic matching*

We use this simple problem to discuss the kinds of failure that may occur if Fraenkel's warning is ignored. As mentioned in §1, matching should not only replace the missing conditions in the inner and outer problems, but also let us check the correctness of the gauge functions that we have assumed. In other words, it should tell us whether or not any switchback terms exist. If Fraenkel's warning is ignored, the asymptotic matching principle may fail us in both ways.

First, it is reasonable to guess, as James (1974) did, that the outer expansion has the form

$$Z(R,\epsilon) = Z_1(R) + \epsilon^2 Z_2(R) + \dots.$$

Then if we apply the asymptotic matching principle (2.1) with $n=2$, $m=3$ and

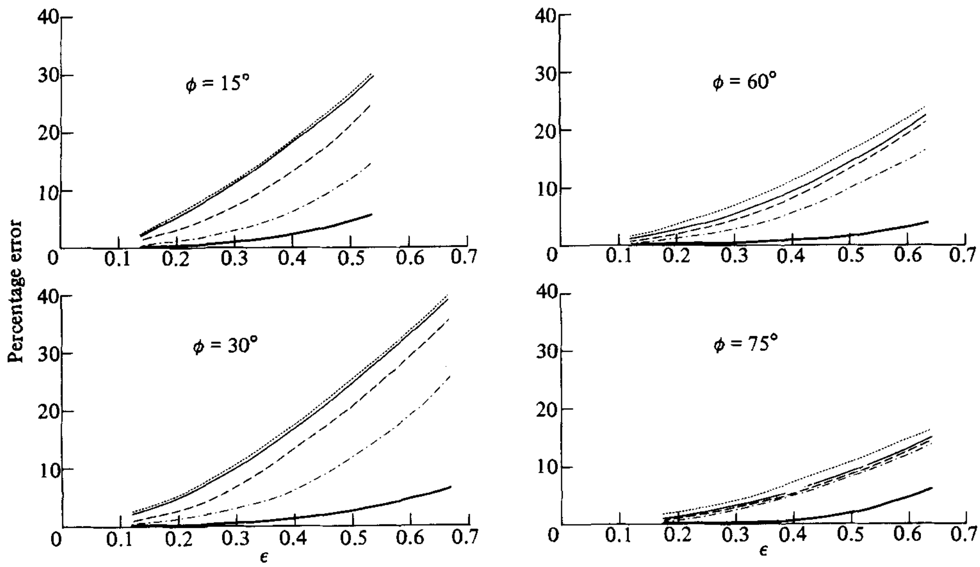

FIGURE 2. Percentage error on the maximum height of the meniscus: ——, first approximation; – – –, second approximation including terms up to $\epsilon^2 \ln^2 \epsilon$; –·–, up to $\epsilon^2 \ln \epsilon$; ——, up to ϵ^2 (full second approximation); ......, James' erroneous second approximation including terms up to $\epsilon^2 \ln^2 \epsilon$.

$f(\epsilon) = 1$, keeping the ϵ^2 term in the outer expansion and the $\epsilon^2 \ln^2 \epsilon$ term in the inner expansion:

$$I_3 O_2 Z(R) = O_2 I_3 z(r).$$

We obtain an incorrect integration constant

$$C_3 = -\tfrac{1}{4}c^3$$

and also appear to confirm the erroneous form of the outer expansion. This shows that the asymptotic matching principle may fail to detect an incorrect form of the series, and that will lead to an erroneous integration constant.

Secondly, if we use the correct form of the outer series

$$Z(R,\epsilon) = Z_1(R) + \epsilon^2 \ln \epsilon\, Z_2^*(R) + \epsilon^2 Z_2(R) + \ldots,$$

but apply the asymptotic matching principle (2.1) with $n = 2$, $m = 4$ and $f(\epsilon) = 1$, keeping the $\epsilon^2 \ln \epsilon$ term in the outer expansion and the $\epsilon^2 \ln \epsilon$ term in the inner expansion,

$$I_4 O_2 Z(R) = O_2 I_4 z(r),$$

we obtain the erroneous results

$$C_3' = -[\tfrac{1}{2}c - \tfrac{3}{4}c^3],$$

$$C_3 = \tfrac{1}{2}c - \tfrac{3}{4}c^3,$$

instead of the correct results in (3.20). This shows that, even though the forms of the inner and outer series are both correct, the asymptotic matching principle still can fail us. This is what might happen in the diffraction problem discussed by Crighton & Leppington (1973).

4. Discussion

The diffraction problem of Crighton & Leppington (1973) and our present correction of the meniscus problem suggest that Fraenkel's warning must be heeded whenever the form of a matched expansion is a mixture of powers and logarithms. However, we show in this section how we can partially relax Fraenkel's restriction. Then we conclude by using that result to show that the last known term, of order $Re^3 \ln Re$, in the series for drag of a sphere at low Reynolds number is correct, although it was calculated disregarding Fraenkel's warning.

4.1. *A modified asymptotic matching principle*

Fraenkel's matching principle generates a complete set of algebraic equations for the constants that must be determined. Those result from equating like functions of ϵ and like functions of the independent variable(s) in the problem. However, as far as an individual constant is concerned, only a few equations among the whole set will be needed for finding it. If Fraenkel's warning is violated, the asymptotic matching principle may generate equations that don't belong to the complete set. In other words, the danger in using asymptotic matching principle (2.1) is that it may generate wrong equations.

The reason is that matching 'mixes' terms with different gauge functions. Consider, for example, the left side of Fraenkel's principle (2.2). The operator Π_1 first truncates the asymptotic series for $z(r)$ at the term with gauge function ϵ^1. Next, in order to apply the operator Θ_k we must rewrite in the outer variable $R = \epsilon r$ and expand for small ϵ. In this process a function $r^i \ln^j r$ is replaced by $(R/\epsilon)^j (\ln R - \ln \epsilon)^j$ and expanded, and is accordingly mixed with terms that originally had different gauge functions. Then the operator Θ_k truncates the new series at the term with gauge function ϵ^k. The right-hand side of (2.2) is mixed in a similar way.

Now suppose we choose to do the final equating with the independent variable r (the inner variable). For that purpose we must rewrite the left-hand side once again in terms of $r = R/\epsilon$, yielding what we may denote by a subscript as $[\Theta_k \Pi_1 z(r,\epsilon)]_r$. In this process the mixing is undone; each term regains its original gauge function of the inner expansion. Thus the left-hand side of (2.2) emerges unmixed. It suggests that one can replace the operator Π_1 with I_n and pick out the contribution from the inner expansion to the first n equations by equating like function ϵ in (2.2). Even though the unmixing of terms with different gauge functions doesn't occur in the right-hand side, the choice of using inner variable r on the final equating also allows us to pick out the contribution from the outer expansion to the first n equations by replacing Π_1 with I_n. So, one can successfully get the first n equations from Fraenkel's matching principle by

$$[\Theta_k \mathrm{I}_n z(r,\epsilon)]_r = \left[\mathrm{I}_n \Theta_k \frac{Z(R,\epsilon)}{f(\epsilon)}\right]_r. \tag{4.1a}$$

Similarly, if the final equating is done in the outer variable R, we may use

$$[\mathrm{O}_n \Pi_l \, z(r,\epsilon)]_R = \left[\Pi_l \mathrm{O}_n \frac{Z(R,\epsilon)}{f(\epsilon)}\right]_R. \tag{4.1b}$$

The matching (4.1 a, b) is a mixture of Fraenkel's restricted matching principle (2.2) and the asymptotic matching principle (2.1). The advantage in using (4.1) is that one can obtain a correct subset of equations by stopping at any term in the inner (or outer) expansion so long as all terms of order ϵ^k (or ϵ^l) are kept in the outer (or inner) expansion, and the final equating is done with the inner (or outer) variable. However, that subset may not suffice to determine the required constants, in which case we can try increasing n by one.

For example, if the matching principle (4.1 a) is used in the meniscus problem, and if, like James, we are interested in the height of the meniscus only up to $\epsilon^2 \ln^2 \epsilon$, we can use

$$[\mathrm{I}_3 \Theta_2 Z(R, \epsilon)]_r = [\Theta_2 \mathrm{I}_3 z(r,\epsilon)]_r,$$

where Θ_2 means truncating the outer series at the term in ϵ^2, and I_3 means truncating the inner series at the term in $\epsilon^2 \ln^2 \epsilon$. However, this yields only the relation

$$-\tfrac{1}{4}c^3 - C_3' = C_3, \quad c = \sin\phi.$$

We need one more equation to solve for both constants C_3' and C_3, so we use instead

$$[\mathrm{I}_4 \Theta_2 Z(R, \epsilon)]_r = [\Theta_2 \mathrm{I}_4 z(r, \epsilon)]_r$$

(I_4 means truncating the inner series at the term in $\epsilon^2 \ln \epsilon$). This gives one more equation

$$C_3' = -\tfrac{1}{2}c + \tfrac{1}{4}c^3$$

and leads to the correct result

$$C_3 = -\tfrac{1}{2}c(c^2 - 1).$$

From this, we see that, as far as the $\epsilon^2 \ln^2 \epsilon$ term is concerned, we need not find the ϵ^2 term in the inner expansion.

However, the matching (4.1) also has its disadvantage. As we mentioned before, matching should also let us check whether the form of the expansion that we assumed is correct or not. Since (4.1) represents only the first n equations of the complete set in Fraenkel's matching (2.2), it may not provide enough information to let us check the gauge functions. In other words, when (4.1) is used, one has to be sure of the form of the expansions in both outer and inner series or one has to include all the possible switchback terms.

4.2. *Drag of a sphere*

In this subsection we will discuss the last known term in the drag on a sphere in low-Reynolds-number flow that was calculated by Chester & Breach (1969). (That is the $Re^3 \ln Re$ term in (1.1).) The inner stream function, which is valid in the region close to the sphere, was found to have the form

$$\psi(r, \mu, Re) = \psi_0(r, \mu) + Re\, \psi_1(r, \mu) + Re^2 \ln Re\, \psi_{2l}(r, \mu) + Re^2\, \psi_2(r, \mu) + Re^3 \ln Re\, \psi_{3l}(r, \mu) + Re^3\, \psi_3(r, \mu) + \dots,$$

in which Re is the Reynolds number of the flow, r is the distance from the centre of the sphere, and μ is the cosine of the angle measured from the direction of the oncoming flow. The outer stream function, which is valid in the region far from the sphere, was found to have the form

$$\Psi(\rho, \mu, Re) = \Psi_0(\rho, \mu) + Re\, \Psi_1(\rho, \mu) + Re^2 \Psi_2(\rho, \mu) + Re^2 \ln Re\, \Psi_{2l}(\rho, \mu) + \dots,$$

in which ρ is the outer variable, related to the inner variable r by $\rho = Re\, r$, and Ψ relates to ψ by $\Psi = Re^2 \psi$. Chester & Breach (1969) calculated the $Re^3 \ln Re$ term in the drag by matching the inner series truncated at the term in Re^3 with the outer series truncated at the term in $Re^3 \ln Re$. According to Fraenkel's warning, the coefficient $\frac{27}{80}$ of the $Re^3 \ln Re$ term in the drag may be incorrect, and nothing more can be said unless the matching includes the Re^3 term in the outer expansion, which is believed to be extremely difficult to calculate. However, using the notation introduced in this paper, we can write the matching rule that Chester & Breach actually used as

$$\left\{\Pi_3 O_4\left[\frac{\Psi}{Re^2}\right]\right\}_\rho = \{O_4\Pi_3\psi\}_\rho$$

(in which the operator O_4 means truncating the series Ψ/Re^2 at the $Re\ln Re$ term). As we showed earlier, if the outer and inner stream functions have the correct gauge functions, this matching actually just picks out the first four equations in the whole set which is generated by Fraenkel's matching principle.

We now consider the possibility of having switchback terms in the outer and inner expansions. Let us assume that the inner expansion has a switchback term of order $Re^3\ln^2 Re$; then the matching tells us that a switchback term of order $Re^3\ln^2 Re$ will be needed in the outer expansion. The inner and outer stream functions will have the following forms:

$$\begin{aligned}\psi(r,\mu,Re) = {} & \psi_0(r,\mu)+Re\,\psi_1(r,\mu)+Re^2\ln Re\,\psi_{2l}(r,\mu)+Re^2\,\psi_2(r,\mu)\\ & +Re^3\ln^2 Re\,\psi^*_{3l}(r,\mu)+Re^3\ln Re\,\psi_{3l}(r,\mu)+Re^3\,\psi_3(r,\mu)+\dots,\\ \Psi(\rho,\mu,Re) = {} & \Psi_0(\rho,\mu)+Re\,\Psi_1(\rho,\mu)+Re^2\,\Psi_2(\rho,\mu)+Re^3\ln^2 Re\,\Psi^*_{3l}(\rho,\mu)\\ & +Re^3\ln Re\,\Psi_{3l}(\rho,\mu)+\dots.\end{aligned}$$

The solutions for all the terms except the two switchback terms are in Chester & Breach (1969). The ψ^*_{3l} will be governed by the same differential equation and has the same boundary condition as ψ_{2l}, so the solution is

$$\psi^*_{3l} = C\left(2r^2-3r-\frac{1}{r}\right)Q_1(\mu).$$

Similarly, the Ψ^*_{3l} will be governed by the same differential equation and has the same boundary condition as Ψ_{3l}, so the solution for Ψ^*_{3l} is

$$\begin{aligned}\Psi_{3l} &= N^*(1+\mu)\{1-\exp[-\tfrac{1}{2}\rho(1-\mu)]\}\\ &= -(N^*\rho-\tfrac{1}{4}N^*\rho^2)\,Q_1(\mu)-\tfrac{1}{4}N^*\rho^2Q_2(\mu)+\dots,\end{aligned}$$

where

$$Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu')\,d\mu'$$

and $P_n(\mu)$ is the Legendre polynomial of degree n. One can perform the matching using

$$\left\{\Pi_3 O_4\left[\frac{\Psi}{Re^2}\right]\right\}_\rho = \{O_4\Pi_3\psi\}_\rho$$

(in which the 4th term in the outer expansion means the $Re\ln^2 Re$ term). Matching in the term $Re\ln^2 Re\,Q_1(\mu)$ gives

$$\tfrac{1}{4}N^* = 2C$$

and

$$N^* = 0,$$

which gives

$$C = 0.$$

This means that the two switchback terms do not exist, and similarly no higher-order switchback terms of order Re^3 will exist. It shows that Chester & Breach (1969) indeed use the correct forms for the inner and outer expansions. Then our previous argument confirms that the coefficient $\frac{27}{80}$ for the term $Re^3 \ln Re$ in the drag is correct.

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